Lecture 8: September 23

An example. The period domains for complex Hodge structures that we are considering here are typically much bigger than the "classical" period domains for rational Hodge structures. Since somebody asked about this last time, let me show you an example. The Hodge structure of a K3-surface looks like

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

with subspaces of dimension $1, 20, 1$. The polarization (constructed from a Kähler form ω as in Lecture 3) is positive definite on $H^{2,0}(X)$ and $H^{0,2}(X)$, and negative definite on $H^{1,1}(X)$. Abstractly, this means that we are looking at Hodge structures on $V = \mathbb{C}^{22}$ of the form

$$V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

with $\dim V^{2,0} = \dim V^{0,1} = 1$, that are polarized by the hermitian pairing

$$h(x,y) = x_1\overline{y_1} + x_2\overline{y_2} - \bigl(x_3\overline{y_3} + \dots + x_{22}\overline{y_{22}}\bigr).$$

The Hodge filtration consists of a line $F^2 \subseteq \mathbb{C}^{22}$ and a hyperplane $F^1 \subseteq \mathbb{C}^{22}$, such that $F^2 \subseteq F^1$. Therefore $\check{D}$ is exactly the incidence variety

$$\bigl\{ (P,H) \in \mathbb{P}^{21} \times (\mathbb{P}^{21})^* \bigm| P \in H \bigr\},$$

and so $\dim \check{D} = 41$. The period domain is an open subset of $\check{D}$, and therefore also of dimension 41. Compare this with the period domain for *rational* Hodge structures of K3-type, which has dimension 20. The reason for the difference is that a rational (or real) polarized Hodge structure of K3-type is determined by F^2 (because F^1 is the orthogonal complement of $V^{0,2}$, which is the complex conjugate of F^2), whereas a complex Hodge structure is not.

Variations of Hodge structure on the punctured disk. We are going to spend the next several weeks talking about polarized variations of Hodge structure on the punctured disk. This is one of the most important topics in Hodge theory, because it serves as a foundation for all the later developments (such as Saito's theory of Hodge modules). All the key results, for polarized variations of *real* Hodge structure, are contained in a long paper by Wilfried Schmid from the early 1970s. I am going to present a different proof that, in my opinion, has two advantages: (1) It works for polarized variations of complex Hodge structure. (2) It shows more clearly what is going on, and why certain new structures (such as the monodromy weight filtration) appear.

Suppose then that we have a polarized variation of Hodge structure of weight n on the punctured disk

$$\Delta^* = \bigl\{ t \in \mathbb{C} \bigm| 0 < |t| < 1 \bigr\}.$$

As usual, I denote the holomorphic vector bundle by $\mathscr{V}$, the connection by ∇, the Hodge bundles by $F^p\mathscr{V}$, and the polarization by $h_{\mathscr{V}} \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^\infty_{\Delta^*}$. In order to compare the Hodge structures at different points, we pull back to the universal covering space

$$\tilde{\mathbb{H}} = \bigl\{ z \in \mathbb{C} \bigm| \operatorname{Re} z < 0 \bigr\} \xrightarrow{\exp} \Delta^*,$$

and consider the vector space of flat sections

$$V = H^0\bigl(\tilde{\mathbb{H}}, \exp^* \mathscr{V}\bigr)^{\exp^* \nabla}.$$

For any two flat sections $v', v'' \in V$, the function $(\exp^* h_{\mathscr{V}})(v', v'')$ is constant on $\tilde{\mathbb{H}}$, and if we denote the constant value by $h(v', v'')$, we obtain a nondegenerate hermitian pairing

$$h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}.$$

Under the trivialization $\mathscr{O}_{\tilde{\mathbb{H}}} \otimes_{\mathbb{C}} V \cong \exp^* \mathscr{V}$, the Hodge bundles $\exp^* F^p\mathscr{V}$ become subbundles of the trivial bundle, and so we get the period mapping

$$\Phi\colon \tilde{\mathbb{H}} \to D,$$

where $D \subseteq \check{D}$ is the period domain parametrizing Hodge structures on V that have weight n and the given Hodge numbers, and are polarized by h. (There is no natural choice of reference point in this case, so just pick any point $o \in D$ as the reference point.) The point $\Phi(z) \in D$ determines a Hodge filtration

$$\cdots \subseteq F^{p+1}_{\Phi(z)} \subseteq F^p_{\Phi(z)} \subseteq \cdots$$

and a Hodge decomposition, for which I will use the notation

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)}.$$

Let us talk about the monodromy action on V. The fundamental group of Δ^* is naturally identified with the group

$$\mathbb{Z}(1) = \left\{ z \in \mathbb{C} \;\middle|\; e^z = 1 \right\} \subseteq \mathbb{C},$$

which acts on the universal covering space $\tilde{\mathbb{H}}$ by translations. (After choosing $i = \sqrt{-1}$, the positive generator is the element $2\pi i$; it corresponds to a positively oriented loop around the origin in Δ^*.) Following Schmid, we define the *monodromy transformation* $T \in \mathrm{GL}(V)$ by the formula

$$v(z - 2\pi i) = (Tv)(z).$$

That is to say, for any flat section $v \in V$, the function $z \mapsto v(z-2\pi i)$ is again a flat section of $\exp^* \mathscr{V}$, and $Tv \in V$ is this flat section. The reason for the minus sign is that, in any covering space, the group of deck transformation naturally acts on the right, and so one has to act by the inverse of a given element to obtain a left action. With this convention, the monodromy transformation is given by parallel transport in the *clockwise* direction around the origin. It is easy to see that $T \in G_{\mathbb{R}}$, where $G_{\mathbb{R}} \subseteq \mathrm{GL}(V)$ is the subgroup preserving the pairing h.

Lemma 8.1. *For every $v', v'' \in V$, we have $h(Tv', Tv'') = h(v', v'')$.*

Proof. By definition, $h(Tv', Tv'')$ is the constant value of the function

$$z \mapsto (\exp^* h_{\mathscr{V}})\big((Tv')(z), (Tv'')(z)\big).$$

Since $(Tv')(z) = v'(z - 2\pi i)$, we get

$$\begin{aligned} h(Tv', Tv'') &= (\exp^* h_{\mathscr{V}})\big(v'(z-2\pi i), v''(z-2\pi i)\big) \\ &= (\exp^* h_{\mathscr{V}})\big(v'(z), v''(z)\big) = h(v', v''). \end{aligned}$$

□

The period mapping $\Phi\colon \tilde{\mathbb{H}} \to D$ has the following additional property:

(8.2) $$\Phi(z + 2\pi i) = T\Phi(z) \quad \text{for } z \in \tilde{\mathbb{H}}$$

It reflects the fact that the variation of Hodge structure is defined on Δ^*. Since this might be a bit confusing at first, let me carefully derive this relation. The trivialization $\mathscr{O}_{\tilde{\mathbb{H}}} \otimes_{\mathbb{C}} V \cong \exp^* \mathscr{V}$ gives us, for every point $z \in \tilde{\mathbb{H}}$, an isomorphism

$$f_z\colon V \to (\exp^* \mathscr{V})\big|_z \cong \mathscr{V}\big|_t, \quad f_z(v) = v(z).$$

Here $t = e^z$. Since $v(z + 2\pi i) = (T^{-1}v)(z)$, the diagram

$$\begin{tikzcd} V \arrow[dr, bend left, "f_{z+2\pi i}"] \arrow[d, "T^{-1}"'] & \\ V \arrow[r, bend right, "f_z"'] & \mathscr{V}\big|_t \end{tikzcd}$$

commutes. The way the period mapping is constructed, we have

$$F^p_{\Phi(z+2\pi i)} = f^{-1}_{z+2\pi i}\Bigl(F^p\mathscr{V}\big|_t\Bigr) = (T^{-1})^{-1} f^{-1}_z\Bigl(F^p\mathscr{V}\big|_t\Bigr) = TF^p_{\Phi(z)},$$

which is exactly (8.2). Together with the distance decreasing property of period mappings, this formula has the following striking consequence, known as the *monodromy theorem.*

Theorem 8.3. *Let T be the monodromy transformation of a polarized variation of Hodge structure on Δ^*. Then all eigenvalues of T have absolute value* 1*.*

Proof. After rescaling h by a constant factor, we can apply Corollary 7.10, which says that for any pair of points $z_1, z_2 \in \tilde{\mathbb{H}}$, one has

$$d_D\Bigl(\Phi(z_1), \Phi(z_2)\Bigr) \leq d_{\tilde{\mathbb{H}}}(z_1, z_2).$$

(Recall that $\tilde{\mathbb{H}}$ is isomorphic to the unit disk.) For any integer $n \geq 1$, the Poincaré distance between the points $-n$ and $-n + 2\pi i$ is roughly $1/n$, hence

$$d_D\Bigl(T\Phi(-n), \Phi(-n)\Bigr) = d_D\Bigl(\Phi(-n+2\pi i), \Phi(-n)\Bigr) \leq \frac{C}{n}$$

for some constant $C \geq 0$. Now choose an element $g_n \in G_{\mathbb{R}}$ with the property that $\Phi(-n) = g_n \cdot o$, where $o \in D$ is the reference point. Since the $G_{\mathbb{R}}$-action on D preserves distances, we get

$$d_D\Bigl(g_n^{-1}Tg_n \cdot o, o\Bigr) = d_D\Bigl(Tg_n \cdot o, g_n \cdot o\Bigr) \leq \frac{C}{n}.$$

Recall that $D \cong G_{\mathbb{R}}/H$, and that the stabilizer H of the point $o \in D$ is a compact subgroup. The relation from above says that the sequence of elements $g_n^{-1}Tg_n \in G_{\mathbb{R}}$ converges, in the quotient $G_{\mathbb{R}}/H$, to the coset of the identity element. Since H is compact, we can pass to a subsequence and arrange that $g_n^{-1}Tg_n$ converges to an element $T' \in H$. Clearly, T and T' have the same eigenvalues. But all eigenvalues of T' have absolute value 1, because H is contained in the unitary group $U = U(V, \langle\,\rangle_o)$. Therefore the same thing must be true for T itself. □

The monodromy theorem was first proved in the geometric case (= for variations of Hodge structure coming from families of compact Kähler manifolds) by Alan Landman; the proof above comes from Schmid's paper, who credits it to Armand Borel. In fact, one can say more in the geometric case.

Corollary 8.4. *If the variation of Hodge structure comes from a family of compact Kähler manifolds over Δ^*, then all eigenvalues of T are roots of unity.*

Proof. Suppose that $\mathscr{V} = \mathscr{O}_{\Delta^*} \otimes_{\mathbb{C}} R^k f_*\mathbb{C}$, where $f\colon X \to \Delta^*$ is a family of compact complex manifolds (whose total space X is Kähler). In this case, we have a locally constant sheaf $R^k f_*\mathbb{Z}$ of finitely generated $\mathbb{Z}$-modules, and if we let $V_{\mathbb{Z}}$ be the space of sections of its pullback to $\tilde{\mathbb{H}}$, then $V = \mathbb{C} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$ and $T \in \operatorname{End}(V_{\mathbb{Z}})$. This shows that all eigenvalues of T are algebraic integers. Now the result follows from Kronecker's theorem: if all conjugates of an algebraic integer have absolute value 1, then the algebraic integer is a root of unity. □

Canonical extensions. The question we are interested in is what happens to the Hodge structures in the variation as $t \to 0$. By pulling back to $\tilde{\mathbb{H}}$, we were able to obtain a family of Hodge structures on a fixed vector space V; but the problem is that we can no longer talk about $t \to 0$ on the halfspace $\tilde{\mathbb{H}}$. Indeed, because of the relation

$$\Phi(z + 2\pi i) = T\Phi(z),$$

the points $\Phi(z)$ have no chance of converging to a limit as $\operatorname{Re} z \to -\infty$. A more natural way to make the Hodge structures on the different fibers of $\mathscr{V}$ comparable would be to work with a trivialization of the bundle $\mathscr{V}$. There is a distinguished family of such trivializations, determined by the connection.

Suppose for a moment that $\mathscr{V}$ is a holomorphic vector bundle on Δ^*, and that

$$\nabla \colon \mathscr{V} \to \Omega^1_{\Delta^*} \otimes_{\mathscr{O}_{\Delta^*}} \mathscr{V}$$

is a connection. There is a distinguished family of holomorphic vector bundles $\tilde{\mathscr{V}}$ on the disk Δ, called *Deligne's canonical extensions*, that extend $\mathscr{V}$, and such that the connection extends to a logarithmic connection

$$\nabla \colon \tilde{\mathscr{V}} \to \Omega^1_{\Delta}(\log 0) \otimes_{\mathscr{O}_{\Delta}} \tilde{\mathscr{V}}.$$

Here $\Omega^1_\Delta(\log 0) = \mathscr{O}_\Delta \frac{dt}{t}$ is the (locally free) sheaf of logarithmic forms. Define the vector space $V = \tilde{\mathscr{V}}\big|_0$ as the fiber of $\tilde{\mathscr{V}}$ over the origin. The logarithmic connection determines an endomorphism

$$R = \operatorname{Res}_0 \nabla \in \operatorname{End}(V),$$

called the *residue* of the connection, as follows. Given $v \in V$, choose a section $s \in H^0(\Delta, \tilde{\mathscr{V}})$ with $s(0) = v$. Write

$$\nabla s = \frac{dt}{t} \otimes s',$$

with $s' \in H^0(\Delta, \tilde{\mathscr{V}})$, and define $R(v) = s'(0) \in V$. Since

$$\nabla(ts) = dt \otimes s + t\nabla(s) = \frac{dt}{t} \otimes (ts + ts'),$$

it is clear that $R(v)$ only depends on $v = s(0)$, and so $R \in \operatorname{End}(V)$ is well-defined.

Now let us suppose in addition that all eigenvalues of the monodromy transformation (which is defined on the space of flat sections of $\exp^* \mathscr{V}$) have absolute value 1. Each eigenvalue can be written as $e^{2\pi i\alpha}$, for $\alpha \in \mathbb{R}$; to make α unique, we should choose a half-open interval $I \subseteq \mathbb{R}$ of length 1, and require that $\alpha \in I$. For example, $I = [0, 1)$ or $I = (0, 1]$ are possible choices.

Theorem 8.5. *Let $(\mathscr{V}, \nabla)$ be a holomorphic vector bundle with connection on Δ^*, such that all eigenvalues of the monodromy transformation have absolute value* 1. *For any half-open interval $I \subseteq \mathbb{R}$ of length* 1, *there is holomorphic vector bundle $\tilde{\mathscr{V}}$ on Δ with a logarithmic connection*

$$\nabla \colon \tilde{\mathscr{V}} \to \Omega^1_{\Delta}(\log 0) \otimes_{\mathscr{O}_{\Delta}} \tilde{\mathscr{V}},$$

with the following two properties:

(a) *The restriction of $(\tilde{\mathscr{V}}, \nabla)$ to Δ^* is isomorphic to $(\mathscr{V}, \nabla)$.*
(b) *All eigenvalues of the residue* $\operatorname{Res}_0 \nabla \in \operatorname{End}(\tilde{\mathscr{V}}\big|_0)$ *lie in I.*

Moreover, $(\tilde{\mathscr{V}}, \nabla)$ are unique up to isomorphism.

Proof. This is a special case of a much more general result by Deligne, which works for any vector bundle with flat connection on the complement of a normal crossing divisor. Since it is enough for our purposes, let me only explain how one constructs $\tilde{\mathscr{V}}$. Let V be the space of flat sections of $\exp^* \mathscr{V}$; we will see in a moment that the fiber of the canonical extension over the origin is V, so this temporary ambiguity in the notation should not cause any problems. For any $v \in V$,

$$v(z + 2\pi i) = (T^{-1}v)(z),$$

by our definition of the monodromy transformation $T \in \mathrm{GL}(V)$. Every eigenvalue of T can be written as $e^{2\pi i\alpha}$ for a unique $\alpha \in I$. One can then find a (unique) endomorphism $R \in \mathrm{End}(V)$, with eigenvalues contained in I, such that

$$T = e^{2\pi i R}.$$

Now consider the function

$$\tilde{s}_v(z) = \bigl(e^{zR}v\bigr)(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}(R^j v)(z),$$

which defines a holomorphic section of the trivial bundle $\mathscr{O}_{\tilde{\mathbb{H}}} \otimes_{\mathbb{C}} V$. Since

$$\tilde{s}_v(z+2\pi i) = \bigl(e^{2\pi i R}e^{zR}v\bigr)(z+2\pi i) = \bigl(Te^{zR}v\bigr)(z+2\pi i) = \bigl(e^{zR}v\bigr)(z) = \tilde{s}_v(z),$$

it is invariant under the action by $\mathbb{Z}(1)$, hence descends to a holomorphic section $s_v \in H^0(\Delta^*, \mathscr{V})$. If we now define $\tilde{\mathscr{V}} = \mathscr{O}_\Delta \otimes_{\mathbb{C}} V$, then we get an isomorphism

$$\tilde{\mathscr{V}}\big|_{\Delta^*} = \mathscr{O}_{\Delta^*} \otimes_{\mathbb{C}} V \cong \mathscr{V} \tag{8.6}$$

by sending $1 \otimes v$ to the section s_v.

It remains to extend the connection. Since v is a flat section of $\exp^* \mathscr{V}$, we have

$$(\exp^* \nabla)\, \tilde{s}_v(z) = dz \otimes (e^{zR}Rv)(z) = dz \otimes \tilde{s}_{Rv}(z),$$

and therefore (on account of the relation $t = e^z$) also

$$\nabla s_v = \frac{dt}{t} \otimes s_{Rv}.$$

If we now define the logarithmic connection on $\tilde{\mathscr{V}} = \mathscr{O}_\Delta \otimes_{\mathbb{C}} V$ by the rule

$$\nabla(f \otimes v) = df \otimes v + f\frac{dt}{t} \otimes Rv,$$

then under the isomorphism in (8.6), this is an extension of the connection on $\mathscr{V}$. It is also clear that $\tilde{\mathscr{V}}\big|_0 = V$, and that $\mathrm{Res}_0 \nabla = R$. □

The canonical extension (for a given interval I) is unique up to isomorphism. The proof above also shows that the canonical extension has a trivialization

$$\mathscr{O} \otimes V \cong \tilde{\mathscr{V}},$$

in which the logarithmic connection takes the particularly simple form

$$\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv,$$

where $V = \tilde{\mathscr{V}}\big|_0$ and $R = \mathrm{Res}_0 \nabla \in \mathrm{End}(V)$. This trivialization depends on the choice of coordinate t on the disk (because the logarithmic form $\frac{dt}{t}$ does of course).